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15.7 - Triple Integrals

Single ✓
Double ✓
Triple : Next up.

Let f be defined on a rectangular box :

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$

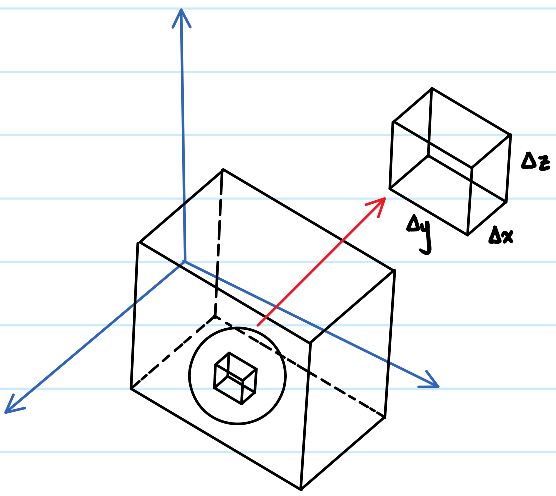
Divide B into sub-boxes by dividing

$[a,b]$	into	l equal subintervals	$[x_i, x_{i+1}]$	of length	Δx.
$[c,d]$	"	m " "	$[y_j, y_{j+1}]$	" "	Δy.
$[r,s]$	"	n " "	$[z_k, z_{k+1}]$	" "	Δz.

- So we divide the rectangular box B into lmn subboxes :

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

and each of these boxes has volume $\Delta V = \Delta x \Delta y \Delta z$.



Then we form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

where,
(x_i^*, y_j^*, z_k^*) is a sample point in B_{ijk}.

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By analogy w/ definition of a double integral, we define the triple integral

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

<u>RmK</u> Rather than an arbitrary sample point, we pick the point (x_i, y_j, z_k).

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows:

<u>Fubini's Theorem for Triple Integrals</u>

If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

<u>Ex</u> Evaluate $\iiint_R (xz - y^3)\,dV$, where R is the rectangular box given by

$$R = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 1\}$$

<u>Soln</u> $$\iiint_E (xz-y^3)\,dV = \int_{-1}^{1}\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx = \int_{-1}^{1}\int_0^2 \left[\frac{xz^2}{2} - y^3 z\right]_0^1 dy\,dx$$

$$= \int_{-1}^{1}\int_0^2 \frac{x}{2} - y^3\,dy\,dx = \int_{-1}^{1}\left[\frac{xy}{2} - \frac{y^4}{4}\right]_0^2 dx = \int_{-1}^{1} x - 4\,dx = \left[\frac{x^2}{2} - 4x\right]_{-1}^{1} = -7$$

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Now we want to define the triple integral over a general region E in three dim'l space (a solid) by the same procedure as double integrals.

• Enclose E into a box rectangular box and define

$$F(x,y,z) = \begin{cases} f(x,y,z) & , (x,y,z) \in E \\ 0 & , (x,y,z) \in B - E \end{cases}$$

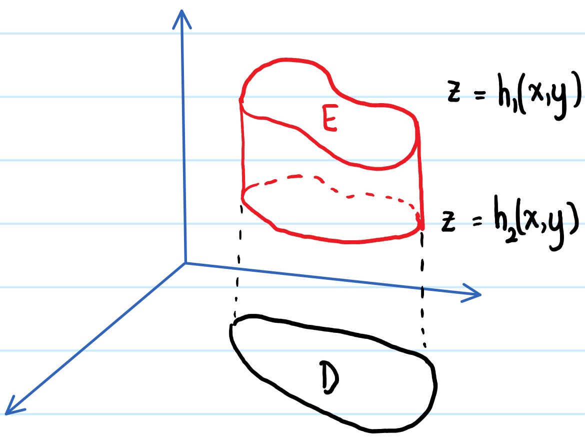
and define $\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$

• The integral exists if f is continuous and boundary of E is "reasonably smooth".

Type 1, 2 and 3 regions

• A solid region E is said to be Type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$E = \{(x,y,z) \mid (x,y) \in D,\ h_1(x,y) \leq z \leq h_2(x,y)\}$, where D is the projection of E onto the xy-plane.



Then if E is a Type 1 region given by

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz \right] dA}$$

treat x and y as a constant and integrate with respect to z.

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Now we have reduced this to a double integral.

Now if D is a type region onto the xy-plane is a type I region, then

$E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}$

and then we get

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

- If D is a Type II region, then

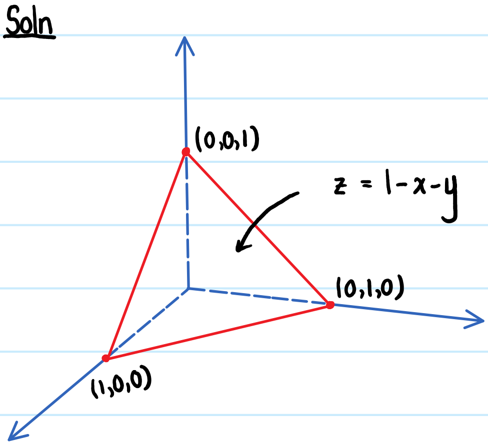
$E = \{(x,y,z) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y),\ h_1(x,y) \le z \le h_2(x,y)\}$

and then we get

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{g_1(y)}^{g_2(y)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

<u>Ex</u> $\iiint_E xy\,dV$ where E is the solid tetrahedron bounded by the four planes $x=0, y=0, z=0$ and $x+y+z=1$.

<u>Soln</u>



Now, we want to set up a triple integral, first draw the solid region E (we are integrating over) and also it's projection onto the xy-plane (provided E is Type 1 region).

- Now we see that $h_1(x,y) = 0$ and $h_2(x,y) = 1-x-y$.

- So we need to figure out what the projection onto the xy-plane is.

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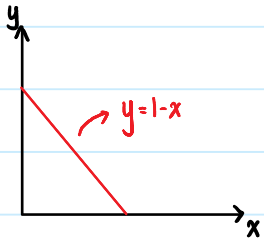


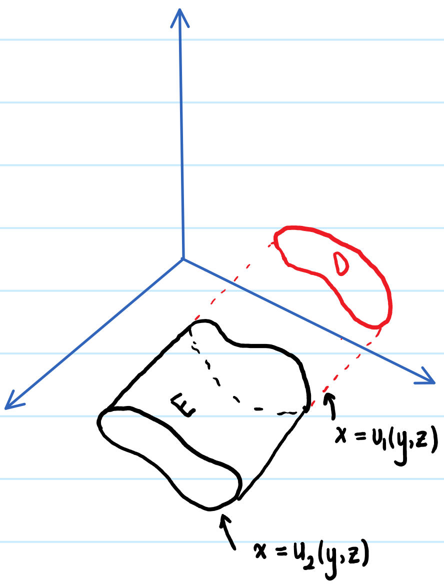
$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$$

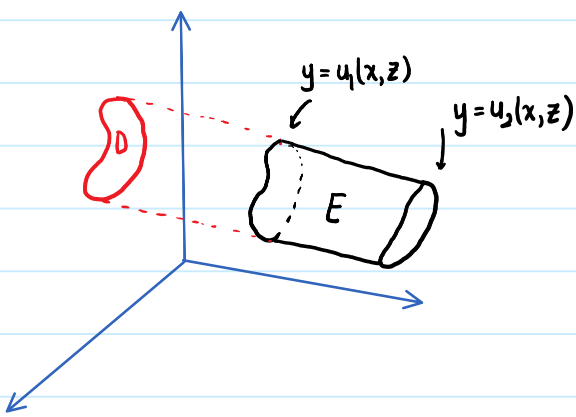
Then, $$\iiint_E xyz\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \Big[xyz \Big]_0^{1-x-y} dy\, dx = \int_0^1 \int_0^{1-x} xy - x^2y - xy^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3} \right]_0^{1-x} dx = \int_0^1 [1-x]^2 \left(\frac{x}{2} - \frac{x^2}{2} - \frac{x(1-x)}{3} \right) dx$$

$$= \int_0^1 -\frac{x^4}{6} + \frac{x^3}{2} - \frac{x^2}{2} + \frac{x}{6}\, dx = \left[\frac{-x^5}{30} + \frac{x^4}{8} - \frac{x^3}{6} + \frac{x^2}{12} \right]_0^1 = \frac{1}{120}.$$

A solid region E is of <u>type 2</u> if it is of the form $E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$





A solid region E is of <u>type 3</u> if it is of the form $E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

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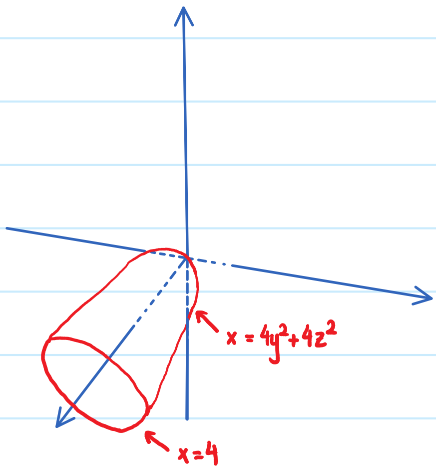
Ex Evaluate $\iiint_E x\,dV$ where E is bounded by the paraboloid $x = 4y^2+4z^2$ and the plane $x = 4$.



The solid E is drawn and we can regard it as a Type 2 solid.

Then we see that it's projection onto the yz-plane is the disc $y^2+z^2 \leq 1$.

Therefore,

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx \right] dA = \frac{1}{2} \iint_D 4^2 - (4y^2+4z^2)^2\,dA$$

The using polar coordinates we obtain

$$= 8 \int_0^{2\pi} \int_0^1 (1-r^4)\,r\,dr\,d\theta = 8 \int_0^{2\pi} d\theta \int_0^1 r - r^5\,dr = 16\pi \left[\frac{r^2}{2} - \frac{r^6}{6} \right]_0^1 = \frac{16\pi}{3}$$

Remark $\iiint_E 1\,dV = V(E)$.

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Density Function and Moments

If the density function of a solid object that occupies the region E is $\rho(x,y,z)$ (mass per unit volume) at any given point (x,y,z), then it's mass is

$$m = \iiint_E \rho(x,y,z)\, dV$$

and it's moment about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)\, dV \,, \quad M_{xz} = \iiint_E y\rho(x,y,z)\, dV \,, \quad M_{xy} = \iiint_E z\rho(x,y,z)\, dV$$

and it's center of mass is located at $(\bar{x},\bar{y},\bar{z})$, where $\bar{x} = \frac{M_{yz}}{m}$, $\bar{y} = \frac{M_{xz}}{m}$, $\bar{z} = \frac{M_{xy}}{m}$.

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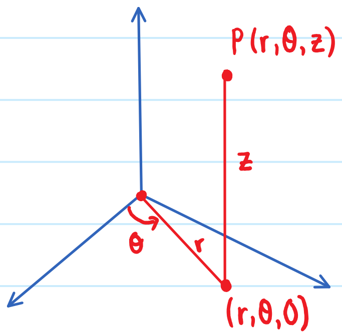
15.8 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates

A point P in three dimensional space is represented by an ordered triple (r, θ, z) where (r, θ) are the polar coordinates of the projection of point P onto the xy-plane and z is the same as usual z-coordinate.



Remark

The cylindrical coordinate system is an extension of polar coordinates to three dimensions.

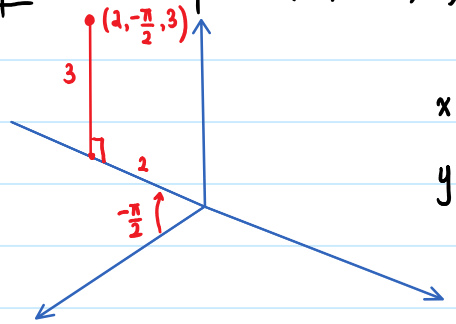
If we have the cylindrical coordinates, the rectangular coordinates can be found using the following equations:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad z = z}$$

Likewise if we have rectangular coordinates, we can find cylindrical coordinates, via

$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z}$$

Example Plot the point $(2, -\pi/2, 3)$, then find the rectangular coordinates of the point.



$x = 2\cos\left(-\frac{\pi}{2}\right) = 0 \qquad z = 3$

$y = 2\sin\left(-\frac{\pi}{2}\right) = -2$

So in rectangular coordinates $(0, -2, 3)$.

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Ex Find the cylindrical coordinates of the point w/ rectangular coordinate $(2,-2,5)$

$$r = \sqrt{(2)^2+(-2)^2} = 2\sqrt{2}$$

$$\tan\theta = \frac{-2}{2} = -1 \Rightarrow \theta = \frac{7\pi}{4} + 2n\pi \quad , z = 3$$

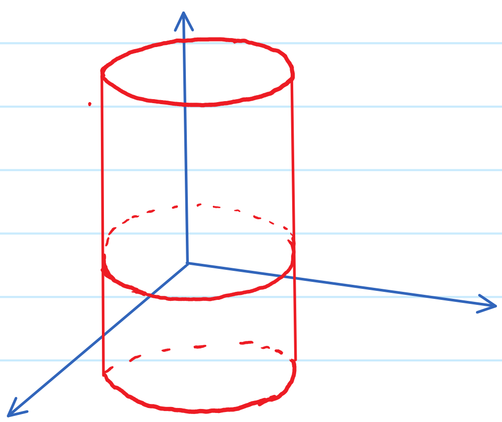
- One set of cylindrical coordinate is $(2\sqrt{2}, 7\pi/4, 5)$ but there are infinitely many choices.

Ex Describe the surfaces whose equation in cylindrical coordinates is

a) $r = 4$

b) $z = r$

a) In two dimensions, $r = 4$ represents a circle of radius 4.
Since there is no restriction on z, this means that z can vary freely.
In particular, for any value of z, we have a circle of radius 5 centered at the z-axis.



In other words, we get a cylinder of radius 5 centered on the z-axis.

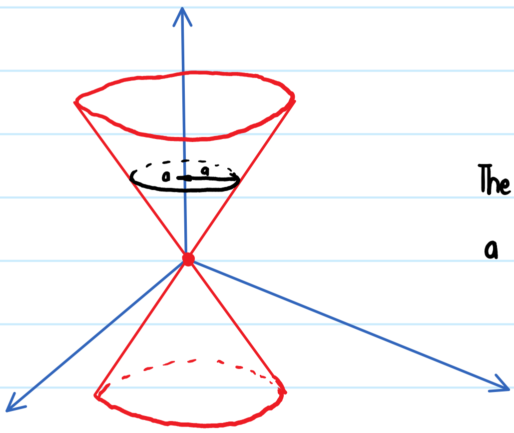
This is the reason behind the name cylindrical coordinates.

Remark Cylindrical coordinates are used when the problem you are working on involves symmetry about an axis. (The cylinder above is symmetric about the z-axis.

b) $z = r$

$$z^2 = r^2 = x^2 + y^2$$

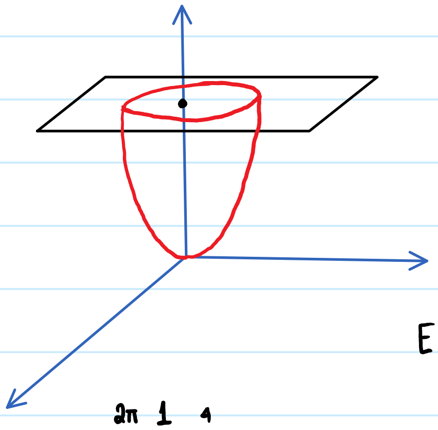
We Know that the graph of $z^2 = x^2 + y^2$ is a cone.



The horizontal trace in the plane $z = a$ is a circle of radius $|a|$.

Ex Find the mass and the center of mass of the solid S bounded by the paraboloid $z = 4x^2+4y^2$ and the plane $z = 4$ if S has constant density K.

Ans



Note that the paraboloid $z = 4x^2+y^2$ intersects the plane when $4 = 4x^2+4y^2$ or $x^2+y^2=1$

Therefore in cylindrical coordinates

$$E = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 4r^2 \le z \le 4\}$$

Then,
$$m = \int_0^{2\pi}\int_0^1\int_{4r^2}^4 Kr\,dr\,dz\,d\theta = K\int_0^{2\pi}\int_0^1 (4r-4r^3)\,dr\,d\theta = K\int_0^{2\pi}\Big[2r^2-r^4\Big]_{r=0}^{r=1} d\theta = K\int_0^{2\pi} d\theta = 2\pi K.$$

- Note that both the density function and the region E are symmetric about the z-axis, $M_{yz} = 0$, $M_{xz} = 0$.

$$M_{xy} = \int_0^{2\pi}\int_0^1\int_{4r^2}^4 zKr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^1 r\Big[\frac{z^2}{2}\Big]_{4r^2}^4 dr\,d\theta = K\int_0^{2\pi}\int_0^1 8r-8r^5\,dr\,d\theta$$

$$= K\int_0^{2\pi}\Big[4r^2-\frac{4}{3}r^6\Big]_0^1 = 2\pi K\cdot\frac{8}{3} = \frac{16\pi K}{3}$$

Finally,
$$(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{8}{3}\right)$$